

The MAJORANA PARTICLES and the MAJORANA SEA

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Abstract

Can one make a Majorana field theory for fermions starting from the zero mass Weyl theory, then adding a mass term as an interaction? The answer to this question is: yes we can. We can proceed similarly to the case of the Dirac massive field theory[1]. In both cases one can start from the zero mass Weyl theory and then add a mass term as an interacting term of massless particles with a constant (external) field. In both cases the interaction gives rise to a field theory for a free massive fermion field. We present the procedure for the creation of a mass term in the case of the Dirac and the Majorana field and we look for a massive field as a superposition of massless fields.

1 Introduction

A Majorana fermion, being its own antifermion, is an unusual particle. If charges should be conserved, the Majorana particle should carry no charge. Among

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bosons the photon is its own antiparticle, that is a "Majorana boson". Among fermions, neutrinos are candidates for Majorana particles.

If you count neutrinos as belonging to the Standard model, the right handed neutrinos appear with no charge and can accordingly interact only through the gravitational force. Going beyond the Standard model, additional quantum numbers, connected with additional gauge fields, can be designed for fermions, and accordingly also for right handed neutrinos. If one starts, for example, with the group $SO(1,13)$ to describe all the internal degrees of freedom (see refs.([2], [3]) of fermions and bosons in an unique way or with the group $SO(10)$ to describe uniquely only charge degrees of freedom, one additional quantum number appears, connected with a gauge field, which is nonzero for left and right handed quarks and leptons, except for charged leptons, and enables the right handed neutrino to interact.

In this talk we shall only present the parallelism between the change of the Dirac (Weyl) sea of massless fermions, for two kinds of interactions, one leading to massive Dirac fermions, the other to massive Majorana fermions. We also shall present for both kinds of the sea the excitations of the sea, representing the physical fermions. This presentation may help to better understand the nature of the Dirac and the Majorana particles. In this talk we pay attention on only spin degrees of freedom. The dimension of the ordinary space-time is four.

2 The zero mass Weyl field theory

We start with massless fermions, described by the Weyl massless fields. We pay attention on a momentum ($p^a = (p^0, \vec{p})$) and a spin of fields. Charges of fields will not be pointed out. The Weyl equation for massless fields

$$2\vec{S} \cdot \vec{p} = \Gamma p^0 \quad (1)$$

determines four states. Here $S^i = \frac{1}{2}\epsilon_{ijk}S^{jk}$, $S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j]$, ($i \in \{1, 2, 3\}$), which are the generators of the Lorentz transformations, determine the spin of states and γ^a , $a \in \{0, 1, 2, 3\}$ are the Dirac operators. The operator $\Gamma = \frac{-i}{3!}\epsilon_{abcd}S^{ab}S^{cd}$ is one of the two Casimir operators of the Lorentz group $SO(1, 3)$ acting in the internal space of spins only and defines the handedness of states. Two eigenstates of Eq.(1) have left handedness ($\langle \Gamma \rangle = r$, $r = -1$), the other two have right handedness¹ ($r = 1$). The left handed solutions have either left helicity

$$h = \frac{2\vec{S} \cdot \vec{p}}{|p^0|}, \quad (2)$$

¹ We shall make use of the symbol Γ for the operator and r (ročnost in slovenian language means handedness) for the corresponding eigenvalue. The symbol h will be used for both, for the helicity operator and for its eigenvalue.

($h = -1$) and positive energy ($p^0 = |p^0|$) or right helicity ($h = 1$) and negative energy ($p^0 = -|p^0|$). The right handed solutions have either right helicity ($h = 1$) and positive energy or left helicity ($h = -1$) and negative energy. We shall denote the positive energy solutions by a symbol u and the negative energy solutions by a symbol v . To determine the positive energy solution completely is enough to tell the momentum \vec{p} , ($p^0 = |\vec{p}|$) and either handedness or helicity: $u_{\vec{p},L} \equiv u_{\vec{p},h=-1}$, $u_{\vec{p},R} \equiv u_{\vec{p},h=1}$, L and R stand for left and right handedness, respectively. Equivalently it follows for the negative energy solution : $v_{\vec{p},L} \equiv v_{\vec{p},h=1}$, $v_{\vec{p},R} \equiv v_{\vec{p},h=-1}$. We shall point out either helicity (h) or handedness (r), depending on what will be more convenient.

After quantizing the field the creation operators are defined, creating the negative energy particles: $d_{\vec{p},L}^{(0)+} \equiv d_{\vec{p},h=1}^{(0)+}$, $d_{\vec{p},R}^{(0)+} \equiv d_{\vec{p},h=-1}^{(0)+}$, and the positive energy particles: $b_{\vec{p},L}^{(0)+} \equiv b_{\vec{p},h=-1}^{(0)+}$, $b_{\vec{p},R}^{(0)+} \equiv b_{\vec{p},h=1}^{(0)+}$. The field operator can then according to ref.[4] be written as:

$$\psi(x) = \sum_{r=\pm 1} \sum_{\vec{p}, p^0=\vec{p}^2} \frac{1}{\sqrt{(2\pi)^3}} (b_{\vec{p},r}^{(0)} u_{\vec{p},r} e^{-ipx} + d_{\vec{p},r}^{(0)} v_{\vec{p},r} e^{ipx}). \quad (3)$$

To simplify the discussions we discretize the momentum and replace the integral with the sum. Then the energy operator $H^{(0)} = \int d^3\vec{x} \psi^\dagger p_0 \psi$ can be written as

$$H^{(0)} = \sum_{r=\pm 1} \sum_{\vec{p}, p^0=\vec{p}^2} |p^0| (b_{\vec{p},r}^{(0)+} b_{\vec{p},r}^{(0)} - d_{\vec{p},r}^{(0)+} d_{\vec{p},r}^{(0)}). \quad (4)$$

If the "totally empty" vacuum state is denoted by $|0\rangle$, then the vacuum state occupied by massless particles up to $\vec{p} = 0$ is (due to discrete values of momenta) equal to

$$|\phi_{(0)}\rangle = \prod_{\vec{p},r} d_{\vec{p},r}^{(0)+} |0\rangle. \quad (5)$$

The energy of such a vacuum state is accordingly $\langle \phi_0 | H^{(0)} | \phi_0 \rangle = \sum_{\vec{p},r} E_{\vec{p},r}^{(0)}$, with $E_{\vec{p},r}^{(0)} = -|\vec{p}|$, which is of course infinite. Accordingly the particle state of momentum \vec{p} , $p^0 = |\vec{p}|$, and handedness r , with the energy, which is for p^0 larger than the energy of the vacuum state, can be written as $b_{\vec{p}}^{(0)+} |\phi_0\rangle$.

3 The charge conjugation

The symmetry operation of charge conjugation is associated with the interchange of particles and antiparticles. Introducing the charge conjugation operator C , with the properties $C^2 = C$, $C^+ = C$, $C\gamma^{a*}C^{-1} = -\gamma^a$, where $(+)$ stays

for hermitian conjugation and (*) for complex conjugation, one finds the charge conjugated field $\psi(x)^C$ to the field $\psi(x)$ as $\psi(x)^C = C\psi(x)$. One accordingly finds for the charge conjugating operator \mathcal{C} , which affects creation and annihilation operators

$$\begin{aligned}\mathcal{C} b_{\vec{p},h=-1}^{(0)+} \mathcal{C}^{-1} &= -d_{-\vec{p},h=1}^{(0)}, & \mathcal{C} b_{\vec{p},h=1}^{(0)+} \mathcal{C}^{-1} &= d_{-\vec{p},h=-1}^{(0)}, \\ \mathcal{C} d_{\vec{p},h=1}^{(0)+} \mathcal{C}^{-1} &= -b_{-\vec{p},h=-1}^{(0)}, & \mathcal{C} d_{\vec{p},h=-1}^{(0)+} \mathcal{C}^{-1} &= b_{-\vec{p},h=1}^{(0)}.\end{aligned}\quad (6)$$

According to section 2 the left handed column concerns the charge conjugation of left handed particles while the right handed column concerns the charge conjugation of right handed particles. One easily finds that the hamiltonian $H^{(0)}$ is invariant under the charge conjugation operation. The charge conjugation operation on the vacuum state $|\phi_{(0)}\rangle = \prod_{\vec{p},r=\pm 1} d_{\vec{p},r}^{(0)+} |0\rangle$ should let it be invariant, since we want it as the physical vacuum state to be charge conjugation invariant. To achieve that we are, however, then forced to let the "totally empty" vacuum state $|0\rangle$ transform under charge conjugation as

$$\mathcal{C}|0\rangle = \prod_{\vec{p},r} (b_{\vec{p},r}^{0\dagger} d_{\vec{p},r}^{0\dagger}) |0\rangle \quad (7)$$

The charge conjugated operator $b_{\vec{p},L}^{(0)+}$ (which generates on a vacuum $|\phi_{(0)}\rangle$ a one particle positive energy state of left helicity) annihilates in the vacuum state $|\phi_{(0)}\rangle$ a negative energy particle state of opposite momentum and helicity and therefore generates a hole, which manifests as an antiparticle. Handedness stays unchanged.

4 The massive Dirac field theory

We shall first treat the case of the massive Dirac field, for which the procedure is well known and simpler than in the massive Majorana case and from which we can learn the procedure. The mass term $\int d^3\vec{x} m_D \bar{\psi}\psi = \int d^3\vec{x} m_D (\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$ can be written, if using the expression for ψ from Eq.(3), as follows

$$H^{(1D)} = m_D \int d^3\vec{x} \bar{\psi}\psi = \sum_{h=\pm 1} \sum_{\vec{p}} H_{\vec{p},h}^{(1D)}, \quad H_{\vec{p},h}^{(1D)} = m_D (b_{\vec{p},h}^{(0)+} d_{\vec{p},h}^{(0)} + d_{\vec{p},h}^{(0)+} b_{\vec{p},h}^{(0)}). \quad (8)$$

If we define

$$\begin{aligned}N_{\vec{p},h} &= h_{\vec{p},h}^+ + h_{\vec{p},h}^-, \\ h_{\vec{p},h}^+ &= b_{\vec{p},h}^{(0)+} b_{\vec{p},h}^{(0)}, \quad h_{\vec{p},h}^- = d_{\vec{p},h}^{(0)+} d_{\vec{p},h}^{(0)},\end{aligned}\quad (9)$$

one easily finds that $[N_{\vec{p},h}, H_{\vec{p},h}^{(0)}] = 0 = [N_{\vec{p},h}, H_{\vec{p},h}^{(1D)}]$. We see that the interaction term $H_{\vec{p},h}^{(1D)}$ does not mix massless states of different helicity. The appropriate basic states, which are eigenstates of the operator for number of particles of definite helicity $N_{\vec{p},h}$ are accordingly defined either with $b_{\vec{p},h=1}^{(0)+}$ and $d_{\vec{p},h=1}^{(0)+}$ with $h = 1$ (but of right and left handedness, respectively) or with $b_{\vec{p},h=-1}^{(0)+}$ and $d_{\vec{p},h=-1}^{(0)+}$ with $h = -1$ (but of left and right handedness, respectively). The first two basic states have $\langle N_{\vec{p},h=1} \rangle = 1$ and $\langle N_{\vec{p},h=-1} \rangle = 0$, while the second two basic states have $\langle N_{\vec{p},h=1} \rangle = 0$ and $\langle N_{\vec{p},h=-1} \rangle = 1$. Diagonalizing $H_{\vec{p},h}^{(D)} = H_{\vec{p},h}^{(0)} + H_{\vec{p},h}^{(1D)}$ within the two basic states of definite helicity (but not handedness), one finds that

$$\begin{aligned} b_{\vec{p},h}^+ &= \alpha_{\vec{p}} b_{\vec{p},h}^{(0)+} + \beta_{\vec{p}} d_{\vec{p},h}^{(0)+}, & p^0 &= |p^0|, \\ d_{\vec{p},h}^+ &= \alpha_{\vec{p}} d_{\vec{p},h}^{(0)+} - \beta_{\vec{p}} b_{\vec{p},h}^{(0)+}, & p^0 &= -|p^0|, \\ \alpha_{\vec{p}} &= \sqrt{\frac{1}{2}(1 + \frac{|\vec{p}|}{|p^0|})}, & \beta_{\vec{p}} &= \sqrt{\frac{1}{2}(1 - \frac{|\vec{p}|}{|p^0|})}, \\ |p^0| &= \sqrt{\vec{p}^2 + m_D^2}. \end{aligned} \quad (10)$$

The operator $b_{\vec{p},h}^+$ creates a massive positive energy one particle state ($p^0 = |p^0|$) and $d_{\vec{p},h}^+$ creates a massive negative energy one particle state ($p^0 = -|p^0|$), both states have momentum \vec{p} and helicity h . Both are eigenstates of the hamiltonian for a massive Dirac field

$$H^{(D)} = \sum_{\vec{p},h} H_{\vec{p},h}^{(D)} = \sum_{\vec{p},h} (H_{\vec{p},h}^{(0)} + H_{\vec{p},h}^{(1D)}) = \sum_{\vec{p},h} |p^0| (b_{\vec{p},h}^+ b_{\vec{p},h} - d_{\vec{p},h}^+ d_{\vec{p},h}) \quad (11)$$

of momentum \vec{p} and helicity h . The state of the Dirac sea of massive particles is now

$$|\phi_{(D)}\rangle = \prod_{\vec{p},h=\pm 1} d_{\vec{p},h}^+ |0\rangle = \pi_\alpha e^{-\sum_{\vec{p},h} \frac{\beta_{\vec{p}}}{\alpha_{\vec{p}}} b_{\vec{p},h}^{(0)+} d_{\vec{p},h}^{(0)}} |\phi_{(0)}\rangle, \quad \pi_\alpha = \prod_{\vec{p}} \alpha_{\vec{p}}. \quad (12)$$

All states up to $p^0 = -m_D$ are occupied and due to Eq.(7) it follows that $\mathcal{C}|\phi_{(D)}\rangle = |\phi_{(D)}\rangle$. The interaction term $H^{(1D)}$ causes the superposition of positive and negative energy massless states (Eq.(10)). One sees that the vacuum state of massive particles can be understood as a coherent state of particle and antiparticle pairs on the massless vacuum state. The energy of the vacuum state of massive Dirac particles $\langle \phi_{(D)} | H^{(D)} | \phi_{(D)} \rangle = \sum_{\vec{p},h} E_{\vec{p},h}^{(D)}$, with $E_{\vec{p},h}^{(D)} = -\sqrt{\vec{p}^2 + m_D^2}$, which is infinite.

According to Eq.(10), the creation and annihilation operators for massive fields go in the limit when $m_D \rightarrow 0$ to the creation and annihilation operators for the massless case.

A one particle state of energy $|p^0| = \sqrt{\vec{p}^2 + m_D^2}$ can be written as $b_{\vec{p},h}^+ |\phi_{(D)}\rangle$, with $b_{\vec{p},h}^+$ defined in Eq.(10). Also this state becomes in the limit $m_D = 0$ a massless Weyl one particle state of positive energy $|\vec{p}|$ above the sea of massless particles.

One easily finds that $H^{(1D)}$ is invariant under charge conjugation and so is therefore also $H^{(D)}$. Taking into account Eqs.(6) it follows

$$\begin{aligned} \mathcal{C} b_{\vec{p},h=-1}^+ \mathcal{C}^{-1} &= -d_{-\vec{p},h=1}, & \mathcal{C} b_{\vec{p},h=1}^+ \mathcal{C}^{-1} &= d_{-\vec{p},h=-1}, \\ \mathcal{C} d_{\vec{p},h=1}^+ \mathcal{C}^{-1} &= -b_{-\vec{p},h=-1}, & \mathcal{C} d_{\vec{p},h=-1}^+ \mathcal{C}^{-1} &= b_{-\vec{p},h=1}. \end{aligned} \quad (13)$$

In the limit $m_D \rightarrow 0$ Eqs. (13) coincide with Eqs. (6). The charge conjugation transforms the particle of a momentum \vec{p} and helicity h into the hole in the Dirac sea of the momentum $-\vec{p}$ and helicity $-h$.

5 The massive Majorana field theory

The Majorana mass term with only left handed fields $m_{ML} \int d^3\vec{x} (\bar{\psi}_L + \bar{\psi}_L^C)(\psi_L + \psi_L^C)$ can be written, if using the expression for ψ from Eq.(3), with the summation going over the left handed fields only and if taking into account the definition of charge conjugation from section 3, as follows

$$H_L^{(1M)} = m_{ML} \int d^3\vec{x} (\bar{\psi}_L + \bar{\psi}_L^C)(\psi_L + \psi_L^C) = \sum_{(\vec{p})^+} H_{\vec{p},L}^{(1M)}, \quad (14)$$

$$H_{\vec{p},L}^{(1M)} = m_{ML} (b_{\vec{p},h=-1}^{(0)+} b_{-\vec{p},h=-1}^{(0)+} + b_{-\vec{p},h=-1}^{(0)} b_{\vec{p},h=-1}^{(0)} + d_{\vec{p},h=1}^{(0)+} d_{-\vec{p},h=1}^{(0)+} + d_{-\vec{p},h=1}^{(0)} d_{\vec{p},h=1}^{(0)}). \quad (15)$$

The symbol $\sum_{(\vec{p})^+}$ means that the sum runs over \vec{p} on such a way that \vec{p} and $-\vec{p}$ is counted only once. Comparing the Majorana interaction term $H^{(1M)}_{\vec{p},L}$ of Eq.(15) with the Dirac interaction term of Eq.(8), one sees that in both cases momentum \vec{p} is conserved as it should be. In Eq. (15) the two creation operators appear with opposite momentum, while in Eq.(8) the creation and annihilation operators appear with the same momentum. Because of that we could pay attention in the Dirac case to a momentum \vec{p} , without connecting \vec{p} with $-\vec{p}$, while in the Majorana case we have to treat \vec{p} and $-\vec{p}$ at the same time.

The Majorana mass term of only right handed fields follows from the mass term of only left handed fields of Eq.(15) if we exchange $h = -1$ with $h = 1$ and $h = 1$

with $h = -1$. We shall treat here the left handed fields only. The corresponding expressions for the massive Majorana right handed fields can be obtained from the left handed ones by the above mentioned exchange of helicities of fields.

It is easy to check that the charge conjugation operator \mathcal{C} from Eq. (6) leaves the interaction term of Eq.(15) unchanged. Accordingly also the hamiltonian

$$H_{\vec{p},L}^{(M)} = H_{\vec{p},L}^{(0)} + H_{\vec{p},L}^{(1M)} \quad (16)$$

is invariant under the charge conjugation: $[H_{\vec{p},L}^{(M)}, \mathcal{C}] = 0$.

As in the Dirac massive case, it is meaningful to use the operators $h_{\vec{p},h=-1}^+ = b_{\vec{p},h=-1}^{(0)+} b_{\vec{p},h=-1}^{(0)}$ and $h_{\vec{p},h=1}^- = d_{\vec{p},h=1}^{(0)+} d_{\vec{p},h=1}^{(0)}$, to choose the appropriate basis within which we shall diagonalize the hamiltonian of Eq.(16). One can check that the operators

$$h_{\vec{p}}^+ = h_{\vec{p},h=-1}^+ - h_{-\vec{p},h=-1}^+, \quad h_{\vec{p}}^- = h_{\vec{p},h=1}^- - h_{-\vec{p},h=1}^-, \quad (17)$$

which count the momentum of states, commute with the hamiltonian of Eq.(16)

$$[h_{\vec{p}}^\pm, H_{\vec{p},L}^{(M)}] = 0. \quad (18)$$

Since basic states, appropriate to describe the vacuum state, should have momentum equal to zero to guarantee the zero momentum of the vacuum, one looks for the basic states with $\langle h_{\vec{p}}^\pm \rangle = 0$. One finds four such states

$$\begin{aligned} |1\rangle &= b_{\vec{p},h=-1}^{(0)+} b_{-\vec{p},h=-1}^{(0)+} |0\rangle, \\ |2\rangle &= \frac{1}{\sqrt{2}} (1 + b_{\vec{p},h=-1}^{(0)+} b_{-\vec{p},h=-1}^{(0)+} d_{\vec{p},h=1}^{(0)+} d_{-\vec{p},h=1}^{(0)+}) |0\rangle, \\ |3\rangle &= d_{\vec{p},h=1}^{(0)+} d_{-\vec{p},h=1}^{(0)+} |0\rangle \end{aligned} \quad (19)$$

$$|4\rangle = \frac{1}{\sqrt{2}} (1 - b_{\vec{p},h=-1}^{(0)+} b_{-\vec{p},h=-1}^{(0)+} d_{\vec{p},h=1}^{(0)+} d_{-\vec{p},h=1}^{(0)+}) |0\rangle. \quad (20)$$

One finds that the state $|4\rangle$ is the eigenstate of the hamiltonian of Eq.(16) with the eigenvalue zero. The hamiltonian applied on first three basic states defines a matrix

$$\begin{pmatrix} 2|\vec{p}| & \sqrt{2}m & 0 \\ \sqrt{2}m & 0 & \sqrt{2}m \\ 0 & \sqrt{2}m & -2|\vec{p}| \end{pmatrix}.$$

Diagonalizing this matrix one finds three vectors and three eigenvalues. The only candidate for the vacuum state is the state $\beta_{\vec{p}}^2 |1\rangle + (-)\sqrt{2} \alpha_{\vec{p}} \beta_{\vec{p}} |2\rangle + \alpha_{\vec{p}}^2 |3\rangle$, with $\alpha_{\vec{p}}$ and $\beta_{\vec{p}}$ defined in Eq.(10), with the eigenvalue $-2\sqrt{|\vec{p}|^2 + m_{ML}^2}$ which

corresponds to the vacuum state of the $-2\sqrt{|\vec{p}|^2 + m_D^2}$ energy in the Dirac massive case $d_{\vec{p},h=1}^+ d_{-\vec{p},h=1}^+ |0\rangle$. The Majorana vacuum state is accordingly

$$|\phi_{(ML)}\rangle = \prod_{(\vec{p})^+} |\phi_{M\vec{p},L}\rangle, \quad |\phi_{M\vec{p},L}\rangle = (\beta_{\vec{p}}^2 |1\rangle + (-)\sqrt{2} \alpha_{\vec{p}} \beta_{\vec{p}} |2\rangle + \alpha_{\vec{p}}^2 |3\rangle), \quad (21)$$

again with $\alpha_{\vec{p}}$ and $\beta_{\vec{p}}$ defined in Eq.(10).

As in the case of the Dirac sea, the Majorana sea can also be written as an exponential operator working on a massless vacuum state

$$|\phi_{(ML)}\rangle = \pi_{\alpha^+}^2 e^{-\sum_{(\vec{p})^+} \frac{\beta_{\vec{p}}}{\alpha_{\vec{p}}} b_{\vec{p},h=-1}^{(0)+} b_{-\vec{p},h=-1}^{(0)+}} e^{-\sum_{\vec{p}} \frac{\beta_{\vec{p}}}{\alpha_{\vec{p}}} d_{\vec{p},h=1}^{(0)} d_{-\vec{p},h=1}^{(0)}} |\phi_{(0)'}\rangle, \quad \pi_{\alpha^+}^2 = \prod_{(\vec{p})^+} \alpha_{\vec{p}}^2,$$

if the massless vacuum state of only left handed particles is written as $|\phi_{(0)'}\rangle = \prod_{(\vec{p},L)} d_{\vec{p},h=1}^{(0)+} d_{-\vec{p},h=1}^{(0)+}$.

Compared to the Dirac particle case it should be noted that for the Majorana case we had to combine both \vec{p} and $-\vec{p}$ when constructing the ground state (Eq.(21)). The energy of the vacuum state of the Majorana left handed particles is $\langle \phi_{(ML)} | H^{(M)} | \phi_{(ML)} \rangle = \sum_{(\vec{p})^+, L} E_{\vec{p},L}^{(ML)}$, with $E_{\vec{p},L}^{(ML)} = -2\sqrt{\vec{p}^2 + m_{ML}^2}$, which is the energy of two majorana particles of momentum $p^a = (-|p^0|, \vec{p})$ and $p^a = (-|p^0|, -\vec{p})$, respectively. The energy of the Majorana sea is again infinite. In the limit $m_{ML} \rightarrow 0$ the Majorana sea becomes the sea of Weyl particles of only left handedness.

Concerning charge conjugation we see that with the somewhat complicated transformation of the "totally empty" vacuum (Eq.(7)) the Majorana physical vacuum $|\phi_{ML}\rangle$ is charge conjugation invariant

$$\mathcal{C}|\phi_{ML}\rangle = |\phi_{ML}\rangle. \quad (22)$$

We have further to construct the one particle Majorana state to generate a physical fermion. The one particle Majorana states can be constructed as superpositions of states with $\langle h_{\vec{p}}^{(+)} + h_{\vec{p}}^{(-)} \rangle = \pm 1$. One finds four times two states which fulfil this condition

$$\begin{aligned} |5\rangle &= b_{\vec{p},h=-1}^{(0)+} |0\rangle, \\ |6\rangle &= b_{\vec{p},h=-1}^{(0)+} d_{\vec{p},h=1}^{(0)+} d_{-\vec{p},h=1}^{(0)+} |0\rangle, \\ \hline |7\rangle &= d_{\vec{p},h=1}^{(0)+} |0\rangle, \\ |8\rangle &= d_{\vec{p},h=1}^{(0)+} b_{\vec{p},h=-1}^{(0)+} b_{-\vec{p},h=-1}^{(0)+} |0\rangle, \\ \hline \end{aligned}$$

$$\begin{aligned}
|9\rangle &= b_{-\vec{p},h=-1}^{(0)+} |0\rangle, \\
|10\rangle &= b_{-\vec{p},h=-1}^{(0)+} d_{\vec{p},h=1}^{(0)+} d_{-\vec{p},h=1}^{(0)+} |0\rangle,
\end{aligned}

$$\begin{aligned}
|5\rangle &= d_{-\vec{p},h=1}^{(0)+} |0\rangle, \\
|6\rangle &= d_{-\vec{p},h=-1}^{(0)+} b_{\vec{p},h=-1}^{(0)+} b_{-\vec{p},h=-1}^{(0)+} |0\rangle,
\end{aligned}
\tag{23}$$

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The first four states have a momentum \vec{p} and the last four states a momentum $-\vec{p}$. The hamiltonian $H_{\vec{p},L}^{(M)}$ defines on these states the block diagonal four two by two matrices. The candidates for the states describing a one particle state of momentum \vec{p} on a vacuum states $|0\rangle$ are states with energy which is for $p^0 = \sqrt{\vec{p}^2 + m_{ML}^2}$ higher than the vacuum state. One finds the corresponding operators

$$\begin{aligned}
b_{\vec{p},h=-1}^+ &= -\beta_{\vec{p}} b_{\vec{p},h=-1}^{(0)+} + \alpha_{\vec{p}} b_{\vec{p},h=-1}^{(0)+} d_{\vec{p},h=1}^{(0)+} d_{-\vec{p},h=1}^{(0)+}, \\
d_{\vec{p},h=1}^+ &= -\alpha_{\vec{p}} d_{\vec{p},h=1}^{(0)+} + \beta_{\vec{p}} d_{\vec{p},h=1}^{(0)+} b_{\vec{p},h=-1}^{(0)+} b_{-\vec{p},h=-1}^{(0)+},
\end{aligned}

$$\begin{aligned}
b_{-\vec{p},h=-1}^+ &= -\beta_{\vec{p}} b_{-\vec{p},h=-1}^{(0)+} + \alpha_{\vec{p}} b_{-\vec{p},h=-1}^{(0)+} d_{\vec{p},h=1}^{(0)+} d_{-\vec{p},h=1}^{(0)+}, \\
d_{-\vec{p},h=1}^+ &= -\alpha_{\vec{p}} d_{-\vec{p},h=1}^{(0)+} + \beta_{\vec{p}} d_{-\vec{p},h=1}^{(0)+} b_{\vec{p},h=-1}^{(0)+} b_{-\vec{p},h=-1}^{(0)+},
\end{aligned}
\tag{24}$$$$

which when applied on a "totally empty" vacuum state $|0\rangle$ generates the one particle states of momentum \vec{p} (the first two operators) and $-\vec{p}$ (the second two operators), respectively.

We would prefer to know, as in the Dirac massive case, the one particle operators which when being applied on a Majorana vacuum state $|\phi_{(ML)}\rangle$ generates a one particle Majorana state with chosen momentum \vec{p} and which commute with the charge conjugate operator \mathcal{C} defined in Eq.(6).

Requiring $B_{\vec{p},h=-1}^+ |\phi_{M\vec{p},L}\rangle = b_{\vec{p},h=-1}^+ |0\rangle$ one finds $B_{\vec{p},h=-1}^+ = \alpha_{\vec{p}} b_{\vec{p},h=-1}^{(0)+} + \beta_{\vec{p}} b_{-\vec{p},h=-1}^{(0)}$, with $\alpha_{\vec{p}}$ and $\beta_{\vec{p}}$ defined in Eq.(10).

Accordingly it follows from the requirement $D_{\vec{p},h=1}^+ |\phi_{M\vec{p},L}\rangle = d_{\vec{p},h=1}^+ |0\rangle$ that $D_{\vec{p},h=1}^+ = \beta_{\vec{p}} d_{\vec{p},h=1}^{(0)+} + \alpha_{\vec{p}} d_{-\vec{p},h=1}^{(0)}$. Taking into account that $\mathcal{C} B_{\vec{p},h=-1}^+ \mathcal{C}^{-1} = -D_{-\vec{p},h=1}^+$ we may conclude that the two operators

$$\mathcal{B}_{\pm\vec{p},h=-1}^+ = \alpha_{\vec{p}} (b_{\pm\vec{p},h=-1}^{(0)+} - d_{\mp\vec{p},h=1}^{(0)}) - \beta_{\vec{p}} (d_{\pm\vec{p},h=1}^{(0)+} - b_{\mp\vec{p},h=-1}^{(0)}), \tag{25}$$

operating on the Majorana vacuum state $|\phi_{ML}\rangle$ generates the one particle Majorana states of momentum $\pm\vec{p}$. It can easily be checked that the Majorana particle is its own antiparticle $\mathcal{C} \mathcal{B}_{\pm\vec{p},h=-1}^+ \mathcal{C}^{-1} = \mathcal{B}_{\pm\vec{p},h=-1}^+$.

In the limit when $m_{ML} \longrightarrow 0$, the operator $\mathcal{B}^+_{\pm\vec{p},h=-1}$ operating on a vacuum state $|\phi_{(ML)}\rangle$, which goes to the vacuum state of the massless case of only left handed particles, gives a state of a Majorana massless particle: $(b^{(0)+}_{\pm\vec{p},h=-1} - d^{(0)}_{\mp\vec{p},h=1}) d^{(0)+}_{\vec{p},h=1} d^{(0)+}_{-\vec{p},h=1} \prod_{\vec{p},\vec{p}\neq\vec{p}} d^{(0)+}_{\vec{p}} |0\rangle$.

One can accordingly find the operators for right handed Majorana particles.

6 Conclusions

We have learned that it is indeed possible to define the Majorana sea in the way the Dirac sea is defined. It stays, however, as an open problem to study how this presentation can be used to better understand the properties of the Majorana particles.

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